Jacobi Method

Consider the system of equation

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad (1)$$

Provided all the diagonal elements are not equal to zero. Now rearranging the equations so that $a_{ii} \neq 0$ i.e.,

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots - a_{1n}x_n) \\ x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots - a_{2n}x_n) \\ \vdots \\ x_n &= \frac{1}{a_n}(b_n - a_{n1}x_1 - a_{n2}x_3 - \cdots - a_{n(n-1)}x_{n-1}) \end{aligned}\right\} \quad (2)$$

Let us assume the first approximations $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$ for the values of $x_1, x_2, \ldots, x_n$ respectively. Substituting these values on R.H.S. of eqn (2), we get a system of second approximation.

$$\left.\begin{aligned} x_1^{(2)} &= \frac{1}{a_{11}}(b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} - \cdots - a_{1n}x_n^{(1)}) \\ x_2^{(2)} &= \frac{1}{a_{22}}(b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)} - \cdots - a_{2n}x_n^{(1)}) \\ \vdots \\ x_n^{(2)} &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1^{(1)} - a_{n1}x_1^{(1)} - \cdots - a_{n(n-1)}x_{n-1}^{(1)}) \end{aligned}\right\} \quad (3)$$

Proceeding in the same way, we obtain third, fourth, etc. approximations. Let $x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)}$.

be the n^{th} approximation, then we get $(n+1)^{th}$ approximation as

$$\left.\begin{aligned} x_1^{n+1} &= \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)} - \dots - a_{1n}x_n^{(n)}\right) \\ x_2^{n+1} &= \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(n)} - a_{23}x_3^{(1)} - \dots - a_{2n}x_n^{(1)}\right) \\ &\vdots \\ x_n^{(n+1)} &= \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^{(n)} - a_{n2}x_2^{(n)} - \dots - a_{n(n-1)}x_{n-1}\right) \end{aligned}\right\} \quad (4)$$

Now, eqn (2) can be written in matrix form as:

$$X = BX + C \qquad \text{—— (5)}$$

Similarly, equation (4) can be written as

$$X^{(n+1)} = BX^{(n)} + C \qquad \text{—— (6)}$$

This method is also known as Method of simultaneous-displacement and this method is sufficient convergent provided $|B| < 1$

Note - In the absence of any better approximation we can take each equal to zero.

Ques Solve the following eqns by Jacobi Method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Soln The given equations can be written as:-

$$\left.\begin{aligned} x &= \frac{1}{27}(85 - 6y + z) \\ y &= \frac{1}{15}(72 - 6x - 2z) \\ z &= \frac{1}{54}(110 - x - y) \end{aligned}\right\} \text{—— (1)}$$

Let us take the first approximation to be $x^{(1)} = 0$, $y^{(1)} = 0$ and $z^{(1)} = 0$, we obtain second approximation from ①

$$x^{(2)} = \frac{1}{27}(85 - 6y^{(1)} + z^{(1)})$$

$$= \frac{1}{27}(85 - 0 + 0) = \frac{85}{27} = 3.15$$

$$y^{(2)} = \frac{1}{15}(72 - 6x^{(1)} - 2z^{(1)})$$

$$= \frac{1}{15}(72 - 0 - 0) = \frac{72}{15} = 4.80$$

$$z^{(2)} = \frac{1}{54}(110 - x^{(1)} - y^{(1)})$$

$$= \frac{1}{54}(110 - 0 - 0) = \frac{110}{54} = 2.04$$

Now we obtain the third approximation as follows :-

$$x^{(3)} = \frac{1}{27}(85 - 6y^{(2)} + z^{(2)})$$

$$= \frac{1}{27}(85 - 6(4.8) + 2.04) = \frac{58.24}{27}$$

$$= 2.16$$

$$y^{(3)} = \frac{1}{15}(72 - 6x^{(2)} - 2z^{(2)})$$

$$= \frac{1}{15}(72 - 6(3.15) - 2(2.04)) = \frac{49.07}{15}$$

$$= 3.27$$

$$z^{(3)} = \frac{1}{54}(110 - x^{(2)} - y^{(2)})$$

$$= \frac{1}{54}(110 - 3.15 - 4.80) = \frac{102.05}{54}$$

$$= 1.89.$$

Now third approximation is :-

$$x^{(4)} = \frac{1}{27}(85 - 6y^{(3)} + z^{(3)}) = \frac{1}{27}(85 - 6(3.27) + 1.89)$$

$$= \frac{67.27}{27} = 2.49$$

$$y^{(4)} = \frac{1}{15}(72 - 6x^{(3)} - 2z^{(3)}) = \frac{1}{15}(72 - 6(2.16) - 2(1.89))$$

$$= \frac{55.26}{15} = 3.68$$

$$z^{(4)} = \frac{1}{54}(110 - x^{(3)} - y^{(3)}) = \frac{1}{54}(110 - 2.16 - 3.27) = \frac{104.57}{54}$$

$$= 1.95$$

The fifth approximation is

$$x^5 = \frac{1}{27}(85 - 6y^{(4)} + z^{(4)}) = \frac{1}{27}(85-6(3.68)+1.95)$$

$$= \frac{64.87}{27} = 2.40$$

$$y^{(5)} = \frac{1}{15}(72 - 6x^{(4)} - 2z^{(4)}).$$

$$= \frac{1}{15}(72 - 6(2.49) - 2(1.95))$$

$$= \frac{53.16}{15} = 3.54$$

$$z^{(5)} = \frac{1}{54}(110 - x^{(4)} - y^{(4)})$$

$$= \frac{1}{54}(110 - 2.49 - 3.68)$$

$$= \frac{103.83}{54} = 1.92$$

Hence, approximate solution is

$x = 2.4$, $y = 3.54$, $z = 1.92$.

Ans